curve gives the solution for every case. Attention is also called to the fact that by this method an angle may be divided in any ratio or ratios in which a straight line may be divided. It is therefore as easily separated into parts having an incommensurable ratio as into any other, provided that that incommensurable ratio can be expressed by means of straight lines.

NOTE ON THE VOLUME OF A TETRAHEDRON IN TERMS OF THE COORDINATES OF THE VERTICES.

By DR. L. E. DICKSON.

1. Quite a variety of propositions of solid analytic geometry are needed for the usual derivation of the volume of a tetrahedron (cf. C. Smith, p. 24). If, as in the present note, we give an elementary proof making use merely of the concept of coordinates, we are in a position to apply the result to derive* easily several of the initial propositions in solid analytics, *e. g.*, that the equation of any plane is of the first degree, and conversely.

The plan of the proof (§ 3) is entirely obvious. The only novelty lies in a certain device which yields the result without computation. This device will first be illustrated in deriving the area of a triangle (§ 2).

2. Let the vertices of a triangle $\triangle$ taken in counter-clockwise order be (x_1, y_1), (x_2, y_2), (x_3, y_3). Then $\triangle$ can be expressed in terms of three right trapezoids with parallel sides y_i. The area of a right trapezoid with parallel sides y_1 and y_2, and base b, is $\frac{1}{2}b(y_1+y_2)$, being half of the rectangle of height y_1+y_2 and base b. Hence

$$2\triangle=(x_1-x_2)(y_1+y_2)+(x_2-x_3)(y_2+y_3)+(x_3-x_1)(y_3+y_1).$$

The device consists in setting $s=y_1+y_2+y_3$. Then

$$2\triangle=(x_1-x_2)(s-y_3)+(x_2-x_3)(s-y_1)+(x_3-x_1)(s-y_2).$$

Since each x occurs once positively and once negatively, the terms in s evidently cancel. The remaining terms give the expansion, according to the second column, of

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

*For plane analytics, this plan is followed in the chapter on graphic algebra in the writer's *College Algebra* (John Wiley and Sons).

3. Consider any tetrahedron $T=P_1P_2P_3P_4$, the notation for the vertices being chosen so that P_1 is above the plane of P_2, P_3, P_4, while the latter lie in counter-clockwise order when viewed from P_1. Denote P_i by (x_i, y_i, z_i); its projection on the x, y-plane is $Q_i=(x_i, y_i, 0)$. Now T can be expressed in terms of four truncated right triangular prisms $P_iP_jP_kQ_iQ_jQ_k$, i, j, k denoting three of the four numbers 1, 2, 3, 4. The area of $Q_iQ_jQ_k$ is given by a determinant (§ 2). Applying the formula (§ 4) for the volume of a truncated right prism, we get

$$6T=D_4(z_1+z_2+z_3)+D_3(z_1+z_2+z_4)+D_2(z_1+z_3+z_4)-D_1(z_2+z_3+z_4),$$

where

$$D_1=\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}, \quad D_2=\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}, \quad D_3=\begin{vmatrix} x_1 & y_1 & 1 \\ x_4 & y_4 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}, \quad D_4=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

The device consists in setting $s=z_1+z_2+z_3+z_4$. Then

$$6T=D_4(s-z_4)+D_3(s-z_3)+D_2(s-z_2)-D_1(s-z_1).$$

Here the terms free of s equal the expansion, according to the third column, of

$$D\equiv\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

The terms multiplying s are derived from the others by replacing each z by -1 and hence equal the expansion of a determinant derived from $-D$ by replacing each z by 1. But a determinant with two columns alike vanishes identically. Hence $T=\frac{1}{6}D$.

4. The volume of a truncated right prism P, whose base is a triangle $\triangle$, and lateral edges are a, b, c, is $\frac{1}{3}(a+b+c)\triangle$. This may be proved as in the geometries, or very simply as follows. Let $a \gtreqless b \gtreqless c$. Let the edge c be DE, d the side of triangle opposite D, h the perpendicular from D to d, so that $\triangle=\frac{1}{2}hd$. The plane through E parallel to $\triangle$ divides P into a right prism of volume $c\triangle$ and a pyramid with summit E, altitude h, and base a trapezoid with parallel sides $a-c$, $b-c$, and common perpendicular d. The area of the trapezoid is $\frac{1}{2}d(a+b-2c)$; the volume of the pyramid is therefore $\frac{1}{3}\triangle(a+b-2c)$. Adding $c\triangle$ to the latter, we get $P=\frac{1}{3}\triangle(a+b+c)$.

To give another proof, extend b (upwards) the length $a-b$. Thus to P we add a triangular pyramid with summit E, and base a right triangle of lege $a-b$ and d, and hence of volume $\frac{1}{2}d(a-b).\frac{1}{3}h=\frac{1}{3}\triangle(a-b)$. Next, extend c the length $a-c$. We thus add on a pyramid of summit E, and base equal to $\triangle$, and hence of volume $\frac{1}{3}\triangle(a-c)$. By these additions, P becomes a right prism of volume $\triangle a$. Hence

$$P+\tfrac{1}{3}\triangle[a-b]+\tfrac{1}{3}\triangle[a-c]=\triangle a, \quad P=\tfrac{1}{3}\triangle[a+b+c].$$

AN EXAMPLE OF THE INDICATRIX IN THE CALCULUS OF VARIATIONS.

By ARNOLD DRESDEN, The University of Chicago.

§ 1. INTRODUCTION.

Suppose there is given a definite integral

$$I=\int_{t_1}^{t_2}F(x,\ y,\ x',\ y')dt, \tag{1}$$

in which x and y are functions of some parameter t, x' and y' are the derivatives of these functions with respect to t. Let the function F be continuous and have continuous derivatives of the first, second and third order in a domain T of the variables x, y, x', y', defined by (x, y) in a region R of the xy-plane, x' and y' finite and restricted by the condition

$$x'^2+y'^2\neq 0. \tag{2}$$

The definite integral (1) may now be taken along an infinitude of curves between $P_1[x(t_1),\ y(t_1)]$ and $P_2[x(t_2),\ y(t_2)]$ in the domain T.

The simplest problem of the Calculus of Variations is to determine among the totality of all these curves, restricted by certain conditions, the one for which I is a *maximum*, or a *minimum**. We shall use the word *extremum* to denote either maximum or minimum.

Let now

$$x=\phi(t),\quad y=\psi(t),\quad t_1\leqq t\leqq t_2, \tag{3}$$

be the equations in parameter-representation† of a curve which minimizes (1). If we restrict ourselves to functions ϕ and ψ, which have continuous first derivatives, the conditions that (3) furnishes a weak‡ minimum for (1) are§

1. *The functions $\phi(t)$ and $\psi(t)$ must satisfy Euler's differential equation, in the Weierstrass-form, i. e.,*

$$\overline{F}_{x'y}-\overline{F}_{xy'}+\overline{F}_1[\phi''\psi'-\phi'\psi'']=0. \tag{I}$$

The function F_1 is defined by

$$F_1(x,\ y,\ x',\ y')=\frac{F_{x'x'}(x,\ y,\ x',\ y')}{y'^2}=-\frac{F_{x'y'}(x,\ y,\ x',\ y')}{x'y'}=\frac{F_{y'y'}(x,\ y,\ x',\ y')}{x'^2}. \tag{4}$$

*For an exact formulation of the problem see O. Bolza, Lectures on the Calculus of Variations, §3.

†See C. Jordan, Cours d'Analyse, Vol. I, 2nd ed., pp. 90-108.

‡See O. Bolza, loc. cit., pp. 69-71.

§Ibid., Chapter IV.

The stroke over the function-symbols is used to denote that the arguments are to be taken as follows:

$$x=\phi(t), \quad y=\psi(t), \quad x'=\phi'(t), \quad y'=\psi'(t).$$

Any curve, satisfying (I), will be called an *extremal.**

II. *Legendre's condition must be fulfilled, i. e.,*

$$\bar{F}_1 \geqq 0, \quad t_1 \leqq t \leqq t_2.$$

III. *Jacobi's condition must be satisfied, i. e.,*

$$t_2 \leq t_1'.$$

t_1' denotes here the parameter-value of the conjugate-point† of $P(t_1)$.

If the minimum shall be strong‡, a fourth condition needs to be satisfied:

IV. $E(x, y; x' y'; \bar{x}', \bar{y}') \geqq 0$, for every point along (3) and for any pair of finite values of $\bar{x}', \bar{y}'$ different from x', y', and restricted by the condition:

$$\bar{x}'^2+\bar{y}'^2 \neq 0.$$

The function E is defined by:

$$E(x, y; x', y'; \bar{x}', \bar{y}') = \bar{x}'[F_{x'}(x, y, \bar{x}', \bar{y}') - F_{x'}(x, y, x', y')] + \\ + \bar{y}'[F_{y'}(x, y, \bar{x}', \bar{y}') - F_{y'}(x, y, x', y')].$$

A stronger form of (IV) is

(IV') $\quad F_1(x, y, \cos\gamma, \sin\gamma) \geqq 0$ along (3), and for $0 \leqq \gamma \leqq 2\pi$.

A curve, satisfying condition (IV'), shall be called a *hyperstrong minimum.*

The conditions for a weak, strong, or hyperstrong maximum are obtained out of (I), (II), (III), (IV), and (IV') by replacing the inequality signs by the opposite ones.

The conditions, as stated above, with the inclusion of equality signs, are the conditions for *improper extrema.*§

By omitting the equality signs from (II), (III), (IV), and (IV'), we obtain the conditions for *proper extrema.*§

If we free ourselves of the restriction, that the functions $\phi'(t)$ and $\psi'(t)$ should be continuous, and admit curves with corners (so called *discon-*

*See O. Bolza, loc. cit., p. 27, p. 123.
†See O. Bolza, loc. cit., p. 60, p. 135.
‡See O. Bolza, loc. cit., p. 69, p. 71.
§See O. Bolza, loc. cit., p. 11.

*tinuous solutions**) as solutions of our problems, still another condition must be satisfied:

(V) *Weierstrass' corner-condition.*† At every corner we must have:

$$\left.\begin{aligned} F_{x'}(x, y, x', y') &= F_{x'}(x, y, \underline{x}', \underline{y}') \\ F_{y'}(x, y, x', y') &= F_{y'}(x, y, \underline{x}', \underline{y}') \end{aligned}\right\}, \tag{V}$$

where x', y', and $\underline{x}'$, $\underline{y}'$ denote the forward and backward derivatives of $\phi(t)$ and $\psi(t)$ at the corner.

2. Carathéodory‡ has given a method by means of which we can decide whether or not the conditions (II), (IV), (IV'), and (V) are satisfied by a given extremal.

For every point of the region R, in which the function F, and its derivatives of first, second, and third order are determined, he defines a curve, called the *Indicatrix*, by means of its equation in polar coordinates:

$$\rho = \frac{1}{F(x, y, \cos\theta, \sin\theta)}, \tag{5}$$

the origin of coordinates being a point G. We make the usual convention of Analytic Geometry, that, if $\rho < 0$, for $\theta = \theta_1$, the absolute value of ρ shall be laid off on the half-line $\theta = \theta_1 + \pi$.

He proves then the following theorems:§

A. If $F(x, y, \cos\theta, \sin\theta)$ is of constant sign for $0 \leqq \theta \leqq 2\pi$, the indicatrix is a closed curve around the origin G.

Proof for A. The theorem follows at once from the definition of the Indicatrix and the usual conventions concerning polar coordinates.

B. If $F(x, y, \cos\theta_0, \sin\theta_0) \gtrless 0$, and the indicatrix has positive curvature at $\theta = \theta_0$,

$$F_1(x, y, \cos\theta_0, \sin\theta_0) \gtrless 0.$$

If $F(x, y, \cos\theta_0, \sin\theta_0) \gtrless 0$, and the indicatrix has negative curvature at $\theta = \theta_0$,

$$F_1(x, y, \cos\theta_0, \sin\theta_0) \lesseqgtr 0.$$

Proof for B. Referring the Indicatrix to rectangular coordinates:

$$\xi = \rho\cos\theta, \quad \eta = \rho\sin\theta,$$

we find

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*See O. Bolza, loc. cit., p. 36.

†Ibid., p. 38, p. 126.

‡C. Caratheodory, *Ueber die discontinuierlichen Loesungen in der Variationsrechnung*, Gottingen, 1904, p. 69; *Mathematische Annalen*, Vol. 62, p. 456.

§*Mathematische Annalen*, Vol. 62, pp. 457, 460, 461, 465.

$$\xi=\frac{\cos\theta}{F(x,\ y,\ \cos\theta,\ \sin\theta)},\qquad \eta=\frac{\sin\theta}{F(x,\ y,\ \cos\theta,\ \sin\theta)}. \tag{6}$$

From this, making use of the formulae:

$$F(x,\ y,\ \cos\theta,\ \sin\theta)=F_x\cos\theta+F_y\sin\theta,$$

$$F'(x,\ y,\ \cos\theta,\ \sin\theta)=-F_x\sin\theta+F_y\cos\theta,^*$$

and the definition of the F_1-function, we obtain:

$$\xi'=-\frac{F_y}{F^2},\qquad \eta'=\frac{F_x}{F^2} \tag{7}$$

and

$$\xi''=\frac{2F'F_y-FF_1\cos\theta}{F^3},\qquad \eta''=-\frac{2F'F_x+FF_1\sin\theta}{F^3},$$

where $'$ indicates differentiation with respect to θ. Consequently:

$$\frac{d^2\xi}{d\eta^2}=\frac{\xi'\eta''-\xi''\eta'}{\xi'^3}=\frac{F_1}{\xi'^2F^3};$$

from which the theorem follows at once.

C. If $F(x,\ y,\ \cos\theta_0,\ \sin\theta_0)\gtrless 0$, and G and $\bar{Q}$ lie on the same side of the tangent to the indicatrix at $Q(\theta_0)$,

$$E(x,\ y;\ \cos\theta_0,\ \sin\theta_0;\ \cos\bar{\theta}_0,\ \sin\bar{\theta}_0)\gtrless 0.$$

If $F(x,\ y,\ \cos\theta_0,\ \sin\theta_0)\gtrless 0$, and G and $\bar{Q}(\bar{\theta}_0)$ lie on opposite sides of the tangent to the indicatrix at $Q(\theta_0)$,

$$E(x,\ y;\ \cos\theta_0,\ \sin\theta_0;\ \cos\bar{\theta}_0,\ \sin\bar{\theta}_0)\lessgtr 0$$

(see Fig. 1).

Proof for C. (See Fig. 1.) From equations (6) and (7) follows

$$F_x X+F_y Y=1, \tag{8}$$

as the equation for the tangent at a point $\theta=\theta_0$ to the Indicatrix for the point $(x_0,\ y_0)$, when X and Y are running coordinates, and the arguments of F_x and F_y are $x_0,\ y_0,\ \cos\theta_0,\ \sin\theta_0$.

For the perpendicular from a point $\bar{Q}(\bar{\theta})$ of the Indicatrix on the tangent at a point $Q(\theta)$, we find:

*See Bolza, loc. cit., p. 120.

$$\overline{Q}M=\frac{-E(x_0, y_0; \cos\theta, \sin\theta; \cos\bar{\delta}, \sin\bar{\delta})}{F(x_0, y_0, \cos\bar{\theta}, \sin\theta)\sqrt{(F^2_{x'}+F^2_{y'})}},$$

while for the perpendicular from the origin G on the same tangent, we obtain

$$GM_G=\frac{-1}{\sqrt{(F^2_{x'}+F^2_{y'})}}<0.$$

From the usual convention concerning the sign of a perpendicular, we obtain then:

$\overline{Q}M<0$, if $\overline{Q}$ and G are on the same side of QM,

$\overline{Q}M>0$, if $\overline{Q}$ and G are on opposite sides of QM,

from which the theorem is at once evident.

D. If the indicatrix for a point (x_0, y_0) admits a double-tangent, touching the curve at the points $A(\alpha)$ and $A'(\alpha')$, the point (x_0, y_0) is the corner of a possible discontinuous solution, the direction of the branches being α and α'.

Proof for D. From the equation for the tangent to the Indicatrix, (8), and from No. (V) of the general theorems (p. 121), we conclude that if the corner-condition is satisfied at a point (x_0, y_0) for two directions α and α', the tangents at the points $A(\alpha)$ and $A(\alpha)$ to the Indicatrix for the point (x_0, y_0) must be coincident, From this, the theorem follows immediately.

After these remarks, we can now go over to the subject proper of this paper, the treatment of a simple problem of the Calculus of Variations, making use of the Indicatrix. The problem treated was given by Caratheadory himself.*

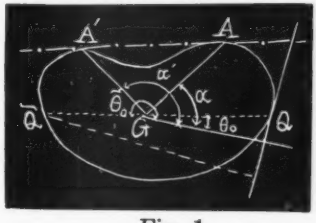

Fig. 1.

3. It is required to minimize the definite integral:

$$I=\int_{t_1}^{t_2}\left[\frac{\sqrt{[x'^2(y^2+1)-2xyx'y'+y'^2(x^2+1)]}}{x^2+y^2+1}-\frac{\sqrt{(x'^2+y'^2)}}{4}\right]dt.$$

We have then:

$$F(x, y, x', y')=\frac{\sqrt{[x'^2(y^2+1)-2xyx'y'+y'^2(x^2+1)]}}{x^2+y^2+1}-\frac{\sqrt{(x'^2+y'^2)}}{4},$$

and find

$$F_{x'y}=F_{xy'}, \quad F_1=\frac{1}{[1+(xy'-x'y)^2]^{\frac{3}{2}}}-\tfrac{1}{4}.$$

*C. Caratheodory, Ueber die disc. Loesungen, etc., p. 38.

(a) Euler's differential equation becomes

$$(x'y''-x''y')F=0.$$

The general integral of this is $x=my+n$, which represents the straight lines of the plane. We obtain also a singular solution from $F_1=0$,

$$1+(xy'-x'y)^2=4^{2/3}, \quad xy'+x'y=(4^{2/3}-1)^{1/2}.$$

We choose now the arc length s, as our functional parameter, and make the following transformation of coordinates (see Fig. 2):

$$\left.\begin{array}{l} x=r\cos\phi, \quad x'=\cos\theta \\ y=r\sin\phi, \quad y'=\sin\theta \\ \psi=\theta-\phi \end{array}\right\}. \tag{9}$$

The singular integral becomes then

$$r\sin\psi=(4^{2/3}-1)^{1/2},$$

also representing a straight line.

The solutions of Euler's differential equation called *extremals* prove to be the straight lines of the plane.

In the sequel, we denote by a, the constant $\sqrt{(4^{2/3}-1)}$.

Applying the transformations (9) to the functions $F(x, y, x', y')$, and $F_1(x, y, x', y')$, we get

$$\left.\begin{array}{l} F=\dfrac{\sqrt{[1+r^2\sin^2\psi]}}{r^2+1}-\frac{1}{4}, \\ \\ F_1=\dfrac{1}{[1+r^2\sin^2\psi]^{\frac{3}{2}}}-\frac{1}{4}. \end{array}\right\} \tag{10}$$

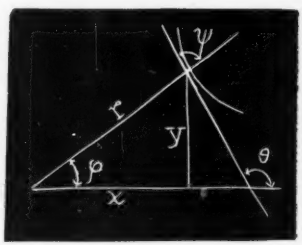

Fig. 2.

(b) For Legendre's condition, we have to consider the sign of F_1. We find from (10 a):

$$F_1>0, \text{ if } r\sin\psi<a, \quad F_1<0, \text{ if } r\sin\psi>a.$$

It follows, that the straight lines, which intersect the circle of radius a (denoted by C_a, see Fig. 3, are minima, and those lying outside C_a are maxima.

(c) The extremals being straight lines, it follows from the geometrical interpretation of the conjugate point,* that Jacobi's condition is fulfilled by every straight line of the plane.

*See Bolza, loc. cit., pp. 60-63, p. 137.

We have then the following result:

I. *Every straight line in the plane intersecting C_a furnishes at least a weak minimum.*

Every straight line in the plane, lying outside C_a, furnishes at least a weak maximum.

The straight lines which are tangent to C_a form a limiting case which will be considered later.

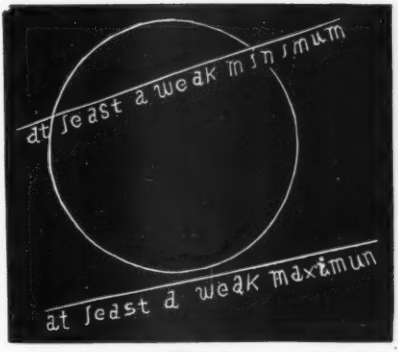

Fig. 3.

(d) We find:

$$E(x,\ y;\ \cos\theta,\ \sin\theta;\ \cos\bar{\theta},\ \sin\bar{\theta}) =$$

$$= \frac{1+r^2\sin^2\bar{\psi}}{\sqrt{[1+r^2\sin^2\bar{\psi}]\ (r^2+1)}} - \frac{\cos(\psi-\bar{\psi}) + r^2\sin.\psi\ \sin.\bar{\psi}}{\sqrt{[1+r^2\sin^2\psi](r^2+1)}} + \frac{\cos(\psi-\bar{\psi})-1}{4},$$

where (7) is applied and $\bar{\psi}=\bar{\theta}-\phi$.

For the further investigation, we have to discuss the sign of this function, which is a very cumbersome problem. At this point, we introduce the Indicatrix for this problem, by means of which the remaining questions can be more readily answered.

4. The *Indicatrix* is determined by the equation:

$$\rho = \frac{1}{\sqrt{\dfrac{(1+r^2\sin^2\psi)}{r^2+1}} - \frac{1}{4}} = \frac{1}{F(r,\ \psi)}. \tag{11}$$

$r \equiv \sqrt{(x^2+y^2)}$ functions here as a parameter, which takes all real positive values, thus furnishing a curve for every point of the plane.

For the character of the Indicatrix it is of importance to know the sign of $F(r,\ \psi)$ and $F_1(r,\ \psi)$ for all values of ψ between 0 and 2π.* Since $\frac{\partial F}{\partial \psi} > 0$, for $0 \leqq \psi \leqq \frac{1}{2}\pi$, we have:

$$F_{\text{min.}} = F(r,\ 0) = \frac{1}{1+r^2} - \frac{1}{4},$$

$$F_{\text{max.}} = F(r,\ \tfrac{1}{2}\pi) = \frac{1}{\sqrt{(1+r^2)}} - \frac{1}{4}.$$

We conclude:

*See page 121, Theorems *A* and *B*.

F *constantly positive*, if $F_{min.} > 0$, *i. e.*, if $\frac{1}{1+r^2} > \frac{1}{4}$ or $r < \sqrt{3}$.

F *constantly negative*, if $F_{max.} < 0$, *i. e.*, if $\frac{1}{\sqrt{(1+r^2)}} < \frac{1}{4}$ or $r < \sqrt{15}$.

F *varying*, if $F_{min.} < 0$, $F_{max.} > 0$, *i. e.*, if $\sqrt{3} < r < \sqrt{15}$.*

We have previously found, that:

$$F_1 > 0, \text{ if } r \sin\psi < a.$$
$$F_1 < 0, \text{ if } r \sin\psi > a.$$

These results show, that the character of the Indicatrix will be essentially different for points lying in one of the four regions, into which the plane is divided by the three circles of radius a, $\sqrt{3}$, and $\sqrt{15}$, respectively (see Fig. 4, circles C_a, C_3, and C_{15}).

The problem is now reduced to the discussion of the properties of the Indicatrix in each of the four cases:

I. $0 < r < a$.
II. $a < r < \sqrt{3}$.
III. $\sqrt{3} < r < \sqrt{15}$.
IV. $\sqrt{15} < r$,

after which the limiting cases $r = a$, $\sqrt{3}$, and $\sqrt{15}$, respectively, still have to be considered.

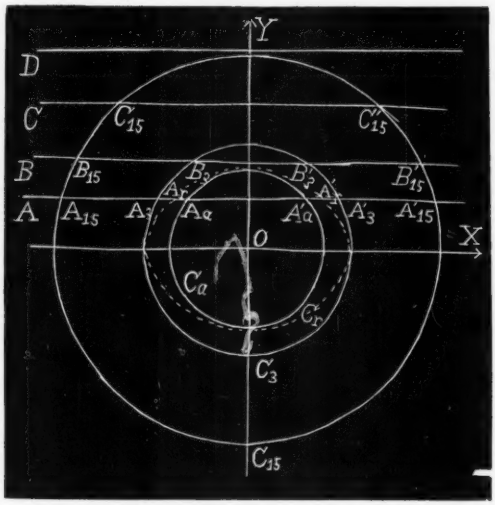

Fig. 4.

(To be continued.)

*$r = \sqrt{3}$ and $r = \sqrt{15}$ are two limiting cases, which will be considered later.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

181. Proposed by A. H. HOLMES, Brunswick, Maine.

Sum the series, $1+2^m+3^m+4^m+\ldots+n^m$.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

An elementary solution by the theory of undetermined coefficients is as follows: Denoting the sum by Sx^m, we put

$$Sx^m = Ax^{m+1}+Bx^m+Cx^{m-1}+Dx^{m-2}+\ldots$$

Since $(x+1)^m = S(x+1)^m - Sx^m$, we have

$$(x+1)^m = A[(x+1)^{m+1}-x^{m+1}] + B[(x+1)^m-x^m] + C[(x+1)^{m-1}-x^{m-1}]+\ldots$$

Developing and arranging, we get

$$x^m+\binom{m}{1}x^{m-1}+\binom{m}{2}x^{m-2}+\binom{m}{3}x^{m-3}+\ldots$$

$$=A\binom{m+1}{1}x^m+\left[A\binom{m+1}{2}+B\binom{m}{1}\right]x^{m-1}+$$

$$\left[A\binom{m+1}{3}+B\binom{m}{2}+C\binom{m-1}{1}\right]x^{m-2}+\ldots+A+B+C+D+\ldots$$

$$\therefore A\binom{m+1}{1}=1,\ A\binom{m+1}{2}+B\binom{m}{1}=\binom{m}{1};$$

$$A\binom{m+1}{3}+B\binom{m}{2}+C\binom{m-1}{1}=\binom{n}{2},\ \text{etc.}$$

$$\therefore A=\frac{1}{m+1},\ B=\tfrac{1}{2},\ C=\frac{m}{12},\ D=0,\ E=\frac{-m(m-1)(m-2)}{240},\ F=0,\ \text{etc.}$$

$$\therefore Sx^m=\frac{x^{m+1}}{m+1}+\tfrac{1}{2}x^m+\tfrac{1}{6}\cdot\tfrac{1}{2}\binom{m}{1}x^{m-1}-\tfrac{1}{30}\cdot\tfrac{1}{4}\binom{m}{3}x^{m-3}+\ldots,\ \text{in which } \tfrac{1}{6},\ \tfrac{1}{30},$$

etc., are Bernoulli's numbers.

Dr. Zerr referred to the solutions of this problem in Todhunter's Algebra, pp. 415-416, and Hall and Knight's Algebra, pp. 336-337.

182. Proposed by O. L. CALLICOTT, Gettysburg, S. Dak.

Find the value of $\sqrt{2}\,\sqrt[3]{2}\,\sqrt[4]{2}\,\sqrt[5]{2}\ldots\sqrt[1000]{2}$.

I. Solution by W. D. LAMBERT, Washington, D. C.

Let $\frac{1}{2}P$ be the required product.

Then $\log_{10}P=\log_{10}2(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots\frac{1}{1000})$.

By the Euler-Bernoulli formula for reducing summation to integration,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{x}=v+\log_e x+\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}\cdots$$

v is Euler's constant=0.57721, 56649...

$\therefore \log_{10}P=0.3010300(0.57721566+6.90775528+0.00050000-0.00000008\ldots)$
$=2.253351.$

$\therefore \frac{1}{2}P=89.603$, the required value.

II. Solution by S. A. COREY, Hiteman, Iowa.

If s is the value sought,

$$\log s=(\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+\tfrac{1}{1000})\log 2. \qquad (1)$$

If $\log(1+x)$ be developed by the formula given in Prize Problem 237, Calculus, we get

$$\log(1+x)=0+\frac{x}{2m}\left[\frac{1}{1+x}+1+2\left(\frac{1}{1+\frac{x}{m}}+\frac{1}{1+\frac{2x}{m}}+\ldots+\frac{1}{1+\frac{(m-1)x}{m}}\right)\right]$$

$$+\frac{B_1 x^2}{m^2.2!}\left[\frac{1}{(1+x)^2}-1\right]-\frac{B_2 x^4}{m^4.4!}\left[\frac{3!}{(1+x)^4}-3!\right]+\text{ etc.} \qquad (2)$$

If, now, x is an integer and m be taken equal to x, we have, as x approaches ∞,

$$\lim_{n=\infty}\log n-\left[\tfrac{1}{2}+\sum_{r=2}^{r=\infty} r^{-1}\right]=-\left[\frac{B_1}{2}-\frac{B_2}{4}+\frac{B_3}{6}\ldots\right]=-b; \qquad (3)$$

whence if C be Euler's constant, .577,215,664,901,5..., $b=C-\frac{1}{2}$. Substituting in (2), transposing, adding $\frac{1}{1000}$ to each member, and reducing, we get, by making $(1+x)=1000$,

$$(\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+\tfrac{1}{1000})=\log 1000-1+\tfrac{1}{2000}+C-\left[\frac{B_1}{2.1000^2}-\frac{B_2}{4.1000^4}+\ldots\right]$$

$=6.485,470,860,55$, or $s=2^{6.485,470,860,55}=89.602,734.8\ldots$

Also solved by G. B. M. Zerr, J. Scheffer, and A. H. Holmes.

GEOMETRY.

312. Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania, Philadelphia, Pa.

A variable circle passes through a fixed point and is tangent to a given circle. If a diameter of the first circle passes through the fixed point find the locus of its other extremity.

Solution by G. W. GREENWOOD, M. A. (Oxon), Roanoke College, Salem, Va.

Since the circle of a radius vector of a central conic is always tangent to its auxiliary circle, the required locus is the conic having the given circle as auxiliary circle and the given point as one focus.

Or, the following method:

Call the center of the given circle 0; its radius, a; the given point, S; the point symmetrical to S with respect to 0, S'; the radius of the circle through S tangent to the given circle, r; the other extremity of the diameter through S, P; the mid-point of SP (the center of the tangent circle), A. Suppose that S is *without* the circle. If the circles are tangent externally, we have

$$PS'-PS=2(AO-AS)=2(AO-r)=2a.$$

If the circles are tangent internally, we get

$$PS-PS'=2a.$$

Therefore, the locus of P is an hyperbola.

Next, suppose that S is *within* the given circle. We get

$$PS'+PS=2(AO+r)=2a.$$

Therefore, the locus of P is an ellipse.

In either case the conic has the points S and S' as foci.

The cases in which S coincides with the center of the given circle, or is on the given circle, may easily be dealt with independently.

Also solved by G. B. M. Zerr, J. Scheffer, C. N. Schmall, and J. S. Brown.

313. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that an algebraic curve of odd degree which is symmetrical with respect to a center has the center on the curve.

No solution has been received.

314. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

Find the area of the triangle bounded by the lines $l\alpha+m\beta+n\gamma=0$; $l'\alpha+m'\beta+n'\gamma=0$; $l''\alpha+m''\beta+n''\gamma=0$, where α stands for $x\cos\alpha+y\sin\alpha-p$, etc. [See Salmon's *Conic Sections*, 6th ed., p. 130, Ex. 1.]

Solution by the PROPOSER.

The formula for the double area of the triangle whose sides are the lines $Ax+By+C=0$, $A'x+B'y+C'=0$, and $A''x+B''y+C''=0$, is

$$\frac{\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix}^2}{\begin{vmatrix} A & B \\ A' & B' \end{vmatrix} \times \begin{vmatrix} A' & B' \\ A'' & B'' \end{vmatrix} \times \begin{vmatrix} A'' & B'' \\ A & B \end{vmatrix}}$$

(V. Salmon's *Conic Sections*, §39.)

For $2\triangle$, the double area of the triangle of reference, the lengths of whose sides are represented by a, b, and c, the formula gives:

$$\frac{\begin{vmatrix} \cos\alpha & \sin\alpha & p \\ \cos\beta & \sin\beta & p' \\ \cos\gamma & \sin\gamma & p'' \end{vmatrix}^2}{\begin{vmatrix} \cos\alpha & \sin\alpha \\ \cos\beta & \sin\beta \end{vmatrix} \times \begin{vmatrix} \cos\beta & \sin\beta \\ \cos\gamma & \sin\gamma \end{vmatrix} \times \begin{vmatrix} \cos\gamma & \sin\gamma \\ \cos\alpha & \sin\alpha \end{vmatrix}}$$

$$\therefore \begin{vmatrix} \cos\alpha & \sin\alpha & p \\ \cos\beta & \sin\beta & p' \\ \cos\gamma & \sin\gamma & p'' \end{vmatrix}^2 = 2\triangle \sin A \sin B \sin C = \frac{2\triangle abc}{8R^3};$$

where $2R=\dfrac{a}{\sin A}$ = diameter of circle circumscribed about triangle of reference. For our problem the double area is

$$\frac{\begin{vmatrix} l\cos\alpha+m\cos\beta+n\cos\gamma & l\sin\alpha+m\sin\beta+n\sin\gamma & lp+mp'+np'' \\ l'\cos\alpha+m'\cos\beta+n'\cos\gamma & l'\sin\alpha+m'\sin\beta+n'\sin\gamma & l'p+m'p'+n'p'' \\ l''\cos\alpha+m''\cos\beta+n''\cos\gamma & l''\sin\alpha+m''\sin\beta+n''\sin\gamma & l''p+m''p'+n''p'' \end{vmatrix}^2}{\begin{vmatrix} l\cos\alpha+m\cos\beta+n\cos\gamma & l\sin\alpha+m\sin\beta+n\sin\gamma \\ l'\cos\alpha+m'\cos\beta+n'\cos\gamma & l'\sin\alpha+m'\sin\beta+n'\sin\gamma \end{vmatrix} \times}$$

$$\times \begin{vmatrix} l'\cos\alpha+\ldots & l'\sin\alpha+\ldots \\ l''\cos\alpha+\ldots & l''\sin\alpha+\ldots \end{vmatrix} \times \begin{vmatrix} l''\cos\alpha+\ldots & l''\sin\alpha+\ldots \\ l\cos\alpha+\ldots & l\sin\alpha+\ldots \end{vmatrix}$$

$$= \frac{\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \sin\alpha & \sin\beta & \sin\gamma \\ p & p' & p'' \end{vmatrix}^2 \times \begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix}^2}{\frac{1}{2R}\begin{vmatrix} a & b & c \\ l & m & n \\ l' & m' & n' \end{vmatrix} \times \frac{1}{2R}\begin{vmatrix} a & b & c \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} \times \frac{1}{2R}\begin{vmatrix} a & b & c \\ l'' & m'' & n'' \\ l & m & n \end{vmatrix}}$$

And the required result is $\dfrac{\triangle abc \begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix}^2}{\begin{vmatrix} a & b & c \\ l & m & n \\ l' & m' & n' \end{vmatrix} \times \begin{vmatrix} a & b & c \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} \times \begin{vmatrix} a & b & c \\ l'' & m'' & n'' \\ l & m & n \end{vmatrix}}$.

Also solved by G. B. M. Zerr, and C. N. Schmall.

CALCULUS.

237. *Prize Problem. Proposed by S. A. COREY, Hiteman, Iowa.

Find an expression for the remainder after n terms in the following development† of $f(a+x)$:

$$f(a+x)=f(a)+\frac{x}{m.2}\left\{f'(a+x)+f'(a)+2\left[f'\left(a+\frac{x}{m}\right)+f'\left(a+\frac{2x}{m}\right)+f'\left(a+\frac{3x}{m}\right)+\ldots+f'\left(a+\frac{m-1}{m}x\right)\right]\right\}-\frac{B_1x^2}{m^2.2!}\left[f''(a+x)-f''(a)\right]+\frac{B_2x^4}{m^4.4!}\left[f^{iv}(a+x)-f^{iv}(a)\right]-\ldots+(-1)^n\frac{B_nx^{2n}}{m^{2n}.(2n)!}\left[f^{(2n)}(a+x)-f^{(2n)}(a)\right]+\ldots,$$

$B_1, B_2, \ldots$ being Bernoulli's numbers.

Solution by the PROPOSER.

Taylor's development for $f(a+x)$ may be written,

$$0=f(a)-f(a+x)+xf'(a)+\frac{x^2}{2!}f''(a)+\frac{x^3}{3!}f'''(a)+\ldots+\frac{x^n}{n!}f^{(n)}(a+\theta_1x), \qquad (1)$$

where $0<\theta_1<1$.

If $f(b-x)$ be similarly developed, and if b be made equal to $(a+x)$ *after* differentiation, we get,

$$0=f(a+x)-f(a)-xf'(a+x)+\frac{x^2}{2!}f''(a+x)-\frac{x^3}{3!}f'''(a+x)+\ldots+(-1)^nf^{(n)}(a+\theta_2x), \qquad (2)$$

where $0<\theta_2<1$.

Adding (1) and (2),

$$0=-x[f'(a+x)-f'(a)]+\frac{x^2}{2!}[f''(a+x)+f''(a)]-\ldots-\frac{x^n}{n!}[f^{(n)}(a+\theta_2x)-f^{(n)}(a+\theta_1x)], \qquad (3)$$

(n odd).

Dividing (3) by x and letting $z_n=[f^{(n)}(a+x)+f^{(n)}(a)]$ and $y_n=[f^{(n)}(a+x)-f^{(n)}(a)]$, we get, by lowering by unity the order of each derivative,

---

*In order to emphasize the importance of finding such an expression for the remainder after n terms as will hold good for all integral values of m and approach 0 as m approaches ∞, and at the same time enable computers to determine absolutely in what cases the development holds, the proposer offers a prize of $10 for the best solution. ED. F.

†See *Annals of Mathematics*, Second Series, Vol. 5, No. 4, July, 1904.

$$0=-y_0+\frac{x}{2!}z_1-\frac{x^2}{3!}y_2+\frac{x^3}{4!}z_3-\frac{x^4}{5!}y_5+\ldots$$

$$-\frac{x^n}{(n+1)!}[f^{(n)}(a+\theta_2 x)-f^{(n)}(a+\theta_1 x)], \qquad (4)$$

(n even).

Now develop $f''(a+x)$ by (4), multiply the result by $-(2x^2/4!)$, and add to (4). In the resulting sum, (5), the term involving z_3 has been eliminated.

$$0=-y_0+\frac{x}{2}z_1-\frac{B_1 x^2}{2!}y_2+\tfrac{5}{115}\cdot\frac{x^4}{4!}y_4-\tfrac{1}{4}\cdot\frac{x^5}{5!}z_5+\tfrac{5}{14}\cdot\frac{x^6}{6!}y_6-\tfrac{11}{24}\cdot\frac{x^7}{7!}z_7+\tfrac{5}{9}\cdot\frac{x^8}{8!}y_8$$

$$-\tfrac{13}{20}\cdot\frac{x^9}{9!}z_9+\ldots+\tfrac{1}{12}\cdot\frac{x^n}{(n-1)!}[f^{(n)}(a+\theta_4 x)-f^{(n)}(a+\theta_3 x)]$$

$$-\frac{x^n}{(n+1)!}[f^{(n)}(a+\theta_2 x)-f^{(n)}(a+\theta_1 x)], \qquad (5)$$

B_1 being Bernoulli's first number, $\frac{1}{6}$.

In a similar manner the term in (5) containing z_5 may be eliminated by developing $f^{iv}(a+x)$ by (4), multiplying result by $+\frac{3x^4}{6!}$ and adding to (5). By continuing this process we may eliminate as many z's as we please, except z_1. Thus to eliminate z_3, z_5, z_7, z_9, and z_{11}, successively, the necessary multipliers are, respectively,

$$-\frac{2x^2}{4!},\ +\frac{3x^4}{6!},\ -\tfrac{20}{3}\cdot\frac{x^6}{8!},\ +\frac{21x^8}{10!},\text{ and }-\frac{90x^{10}}{12!}.$$

The result obtained by eliminating these five z's is found to be identical with the formula given in the problem for the case $m=1$, up to and including the term containing B_5, and may be thus written,

$$0=-y_0+\frac{x}{2!}z_1-\frac{B_1 x^2}{2!}y_2+\frac{B_2 x^4}{4!}y_4-\frac{B_3 x^6}{6!}y_6$$

$$+\frac{B_4 x^8}{8!}y_8-\frac{B_5 x^{10}}{10!}y_{10}+\frac{x^{12}}{12!}R_{12} \qquad (6)$$

where $R_{12}=-12!\left(\frac{\varepsilon_1}{13!}-\frac{2\varepsilon_2}{4!\,11!}+\frac{3\varepsilon_3}{6!\,9!}-\frac{20\varepsilon_4}{3.8!\,7!}+\frac{21\varepsilon_5}{10!\,5!}-\frac{90\varepsilon_6}{12!\,3!}\right)$, (7)

the ε's having some value between the greatest and least value of $[f^{12}(a+\theta_1 x)-f^{12}(a+\theta_2 x)]$.

Let r_{12} be positive and equal to the greatest numerical value which ϵ may take. Then as

$$12!\left(\frac{1}{13!}+\frac{2}{4!\,11!}+\frac{3}{6!\,9!}+\frac{20}{3.8!\,7!}+\frac{21}{10!\,5!}+\frac{90}{12!\,3!}\right)r_{12}=\tfrac{2}{4}\tfrac{7}{5}\tfrac{4}{5}\tfrac{7}{5}\tfrac{8}{5}r_{12},$$

$$-\tfrac{2}{4}\tfrac{7}{5}\tfrac{4}{5}\tfrac{7}{5}\tfrac{8}{5}r_{12}<R_{12}<+\tfrac{2}{4}\tfrac{7}{5}\tfrac{4}{5}\tfrac{7}{5}\tfrac{8}{5}r_{12},$$

where the coefficient of r_{12} is obtained by changing all the negative signs in (7) to positive, substituting unity for each of the ϵ's and adding.

If now

$$f\left(a+\frac{x}{m}\right), f\left[\left(a+\frac{x}{m}\right)+\frac{x}{m}\right], f\left[\left(a+\frac{2x}{m}\right)+\frac{x}{m}\right], \ldots, f\left[\left(a+\frac{m-1}{m}x\right)+\frac{x}{m}\right],$$

be developed successively by (6), simplifying each by the aid of the preceding, the result is identical with the formula given in the problem up to and including B_5, and may be thus written,

$$f(a+x)=f(a)+\frac{x}{m.2}\left[f'(a+x)+f'(a)+2\left(f'\left(a+\frac{x}{m}\right)+f'\left(a+\frac{2x}{m}\right)+\ldots\right.\right.$$

$$\left.\left.+f'\left(a+\frac{m-1}{m}x\right)\right)\right]-\frac{B_1 x^2}{m^2.2!}[f''(a+x)-f''(a)]+\frac{B_2 x^4}{m^4.4!}[f^{iv}(a+x)-f^{iv}(a)]$$

$$-\frac{B_3 x^6}{m^6.6!}[f^{vi}(a+x)-f^{vi}(a)]+\frac{B_4 x^8}{m^8.8!}[f^{viii}(a+x)-f^{viii}(a)]$$

$$-\frac{B_5 x^{10}}{m^{10}.10!}[f^{x}(a+x)-f^{x}(a)]\pm\frac{x^{12}.m.R_{12}}{m^{12}.12!}. \qquad (8)$$

By a close inspection of the method used to eliminate the z_3, z_5, z_7, z_9, and z_{11} from (4), we learn that the following formula may be generally used to obtain the multiplier, $\frac{c_s x^{(s-1)}}{(s+1)!}$, which may be employed to eliminate z_s after all the preceding z's except z_1 have been eliminated, viz.,

$$0=S!\left[\frac{1}{(s+1)!}-\frac{2}{4!\,(s-1)!}+\frac{3}{6!\,(s-3)!}-\tfrac{2}{3}\tfrac{0}{}\cdot\frac{1}{8!\,(s-5)!}\right.$$

$$\left.+\frac{21}{10!\,(s-7)!}-\frac{90}{12!\,(s-9)!}+\ldots+\frac{c_s}{(s+1)!2}\right]. \qquad (9)$$

which must hold for *all odd* values of s except $s=1$. It can also be shown that when s is even, say $2n$, (9) gives the value of $\pm B_n$ according as n is odd or even.

After thus eliminating z_3, z_5, z_7, z_9, ... and z_s from (4), and repeating the process used to obtain (8), we obtain a development which is identical with the development up to and including the term containing B_n where $n=\frac{s-1}{2}$. That this is true for *all* values of n follows from the fact that the two developments become identical when $R=0$ for *any* value of n.

Likewise by a method similar to that used to obtain R_{12}, we may obtain the value of $R_{(2n+2)}$. That method is: Find the sum of all the terms in (9) after making all negative signs positive, and after making $s=(2n+2)$, observing that *no new term is to be added.* Calling this sum $t_{(2n+2)}$ we may write

$$R_{(2n+2)}=\pm t_{(2n+2)}\,[f^{(2n+2)}\,(a+\theta_1 x)-f^{(2n+2)}\,(a+\theta_2 x)],$$

where both θ_1 and θ_2 have some value greater than zero and less than unity, but not the same value unless $R=0$.

The remainder after the term involving B_n may therefore be expressed as follows:

$$\pm\frac{x^{(2n+2)}\,.\,t_{(2n+2)}}{m^{(2n+1)}\,.\,(2n+2)!}[f^{(2n+2)}\,(a+\theta_1 x)-f^{(2n+2)}\,(a+\theta_2 x)],^{*}$$

where the value of $t_{(2n+2)}$ must be computed for higher values of n than are here given, viz.,

$$t_4=\tfrac{8}{15},\ t_6=\tfrac{8}{7},\ t_8=\tfrac{148}{45},\ t_{10}=\tfrac{410}{33},\ t_{12}=\tfrac{27478}{455},\ t_{14}=\tfrac{2340}{9},\ \text{and}\ t_{16}=\tfrac{2410948}{85}.$$

To obtain such higher values of t it is, of course, necessary first to find values of c_s in (9) higher than the following, which were used to find the foregoing values of t, viz.,

$$c_3=-2,\ c_5=+3,\ c_7=-\tfrac{20}{3},\ c_9=+21,\ c_{11}=-90,\ c_{13}=+\tfrac{7601}{15},\ \text{and}\ c_{15}=-3640.$$

The foregoing proof is based on the assumption that all the derivatives are finite and continuous between the limits $x=0$ and $x=x$.

It will be observed that the foregoing solution does not give the value of the remainder after the *general* term as required in the problem, the value of t not being given for more than the first few terms. In actual work the values of t here given are probably all that the computer needs. But, if not, higher values may be found without excessive labor in the manner given. This solution, therefore, places the formula given in the problem on a fairly satisfactory basis in so far as the needs of the computer are involved.

As the Method of Approximation which the writer gave in the June-July, 1906, number of the MONTHLY is based on the formula of the problem,

*The quantitity within the brackets cannot exceed twice the *greatest* variation of $f^{(2n+2)}(a+\theta x)$ for *any single interval* $(x\ m)$ as x increases from $x=0$ to $x=x$.

this solution will serve to determine the limits within which errors in results due to the dropping of higher terms must lie whenever it is possible to determine the limits within which the value of the lowest dropped derivative lies.

NOTE. Mr. Corey informs us that he did not succeed in obtaining a solution until a month after the problem was published, though he had repeatedly attempted to do so. Mr. Corey's solution is quite satisfactory so far as its use for practical computation is concerned, yet greater value would result were an expression found for the remainder after the general term. A prize of $10 is still offered by Mr. Corey for finding an expression for the remainder after the nth term. ED. F.

238. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Solve the differential equations

$$(a)\ (x+y^2)\,dx+(x^2+y)\,dy=0,$$
$$(b)\ (x+xy+y^2)\,dx-(x^2+xy-y)\,dy=0.$$

Solution by GEORGE W. HARTWELL, Columbia University, New York.

$(a)\quad (x+y^2)\,dx+(x^2+y)\,dy=0.$

This equation can be written

$$\frac{dx}{x^2+y}=\frac{dy}{-x-y^2};\ \therefore\frac{dx+dy}{(x-y)\ (x+y-1)}=\frac{xdy+ydx+dx+dy}{(x-y)\,(xy-1)}.$$

Multiplying both members by $x-y$,

$$\frac{dx+dy}{x+y-1}=\frac{xdy+ydx+dx+dy}{xy-1}.$$

Clearing of fractions and transposing,

$$(x+y-1)\,(xdy+ydx)-(xy-1)\,(dx+dy)+(x+y-1)\,(dx+dy)=0.$$

Dividing by $(x+y-1)^2$,

$$\frac{(x+y-1)\ (xdy+ydx)-(xy-1)\ (dx+dy)}{(x+y-1)^2}+\frac{dx+dy}{x+y-1}=0.$$

Integrating, $\frac{xy-1}{x+y-1}+\log(x+y-1)=a$. Multiplying by $(x+y-1)$,

$$xy-1+(x+y-1)\log(x+y-1)=a(x+y-1).$$

$(b)\quad (x+xy+y^2)\,dx-(x^2+xy-y)\,dy=0.$

This equation can be written

$$\frac{dx}{x^2+xy-y}=\frac{dy}{y^2+xy+x};\ \therefore\frac{dx-dy}{(x+y)\ (x-y-1)}=\frac{xdx+ydy}{(x+y)\ (x^2+y^2)}.$$

Multiplying by $(x+y)$, $\frac{dx-dy}{x-y-1}=\frac{xdx+ydy}{x^2+y^2}$. Integrating,

$$2\log(x-y-1)=\log(x^2+y^2)+\log a;\ \therefore(x-y-1)^2=a(x^2+y^2).$$

Also solved by G. B. M. Zerr.

MECHANICS.

198. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Three spheres of the same material, radii R, r, S, rest upon a horizontal plane, touching each other. Find the radius of a sphere of the same material as the others which, being placed upon the other three spheres, will just prevent the latter from separating, the coefficient of friction between the spheres being μ, and between the spheres and the table being μ'.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let x=radius of top sphere center Q, R=radius of bottom sphere center P. The forces acting on the sphere radius R are its own weight W, the friction F, the friction F', and the normal reaction between the two spheres R. Let nW_1 be the part of the weight of the sphere radius x supported by the sphere radius R. $\angle DPC=\beta$.

$$\text{Now } FHC=RDC. \quad F=R\left(\frac{DC}{HC}\right)=R\tan\tfrac{1}{2}\beta. \quad \therefore F=\mu R \text{ or } \mu=\tan\tfrac{1}{2}\beta.$$

$$F'=\mu'[W+nW_1]=R\sin\beta-F\cos\beta=R(\sin\beta-\tan\tfrac{1}{2}\beta\cos\beta]=R\tan\tfrac{1}{2}\beta.$$

$$\therefore R=\frac{\mu'[W+nW_1]}{\tan\frac{1}{2}\beta}=\frac{\mu'}{\mu}[W+nW_1].$$

Resolving vertically, $nW_1=R\cos\beta+F\sin\beta$.

$$\therefore nW_1=R[\cos\beta+\tan\tfrac{1}{2}\beta\sin\beta]=R. \quad \therefore nW_1=\frac{\mu'}{\mu}[W+nW_1].$$

Let δ=density of each sphere. $\quad \therefore W=\delta g\times\tfrac{4}{3}\pi R^3, \quad W_1=\delta g\times\tfrac{4}{3}\pi x^3.$

$$\therefore nx^3=\frac{\mu'}{\mu}[R^3+nx^3]. \quad \therefore n=\frac{\mu'R^3}{[\mu-\mu']x^3}.$$

Let mW_1 be the part of the upper sphere supported by sphere radius r, and pW_1 the part supported by sphere radius S. Then

$$m=\frac{\mu'r^3}{[\mu-\mu']x^3}, \quad p=\frac{\mu'S^3}{[\mu-\mu']x^3}.$$

But $n+m+p=1$.

$$\therefore \frac{\mu'[R^3+r^3+S^3]}{\mu-\mu'}=x^3, \text{ and } \therefore x=\sqrt[3]{\frac{\mu'[R^3+r^3+S^3]}{\mu-\mu'}}.$$

199. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

A sphere of water, radius $\frac{1}{4\,9}$, the earth's radius, is brought together by mutual attractions of particles from a state of infinite diffusion. Find its temperature owing to the amount of work done by these forces.

Solution by W. D. LAMBERT, Washington, D. C.

Let r be the variable radius of the sphere, while forming; a the final radius $=\frac{6.370\times10^8}{49}$ centimeters; ρ the density of water$=1$; k the gravitation constant$=6.665\times10^{-8}$ dynes; J the mechanical equivalent of heat$=4.184\times10^7$ ergs for the centigrade gram-calorie. It is a little easier to conceive the sphere as pulled asunder against its own attraction, and the amount of work will be the same. Suppose the sphere made up of layers, each of thickness dr, and that the sphere has been reduced to radius r. The mass of a layer is $4\pi\rho r^2dr$. The attraction between this mass and the remaining sphere is $\frac{k\times\frac{4}{3}\pi\rho r^3\times4\pi\rho r^2dr}{x^2}$, where x denotes the distance of the layer (supposed to be scattered symmetrically) from the center of the sphere. The work done in removing the layer from the surface of the sphere in question to infinity is $\frac{16}{3}\pi^2k\rho^2r^5dr\int_r^\infty\frac{dx}{x^2}=\frac{16}{3}\pi^2k\rho^2r^4$.

The total work done in removing all layers is

$$\tfrac{16}{3}\pi^2k\rho^2\int_0^a r^4dr=\tfrac{16}{15}\pi^2k\rho^2a^5.$$

Dividing this quantity by the mass and by J we get for the tempera- $\frac{4}{5}\frac{\pi k\rho a^2}{J}$. For substances other than water this result should be multiplied by the specific heat of the substance. Using the numerical values previously given, we get for the temperature $0^\circ.677$ centigrade.

Also solved by G. B. M. Zerr, whose result is 0.656. This difference of result is due to the different values assumed for the constants entering into the solution.

AVERAGE AND PROBABILITY.

183. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A point within a given triangle is joined to each of the corners. What is the average of the sum of the lengths of these three lines?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC be the given triangle, P the random point, A the vertex, BC the base of the triangle, AD the altitude. Through P draw QR, parallel to BC cutting AD in F. Let $AD=p$, $BD=e$, $DC=d$, $AF=x$, $FP=y$. Then $AP=\sqrt{[x^2+y^2]}$. The limits of x are 0 and p; of y, $-QF=ex/p$ and $+FR=dx/p$. Let $M=$average length of AP, $J=$average length of the sum.

$$\therefore M=\int_0^p\int_{-ex/p}^{dx/p}\sqrt{[x^2+y^2]}\,dxdy/\int_0^p\int_{-ex/p}^{dx/p}dxdy$$

$$=\frac{2}{a\,p}\int_0^p \int_{-ex/p}^{dx/p} [x^2+y^2]\,dxdy, \text{ since } d+e=a,$$

$$=\frac{1}{a\,p}\int_0^p \left[\frac{[bd+ce]x^2}{p^2}+x^2\log\left(\frac{d+b}{c-e}\right)\right]dx, \quad \begin{pmatrix} p^2+d^2=b^2 \\ p^2+e^2=c^2 \end{pmatrix}$$

$$=\frac{1}{3a}\left[bd+ce+p^2\log\left(\frac{d+b}{c-e}\right)\right]$$

$$=\frac{1}{3a}\left[b^2\cos C+c^2\cos B+b^2\sin^2 C\log\left(\frac{b[1+\cos C]}{b[1-\cos B]}\right)\right].$$

By similarity,

$$\Delta=\frac{1}{3a}\left[b^2\cos C+c^2\cos B+\frac{4\Delta^2}{a^2}\log\left(\frac{a+b+c}{b+c-a}\right)\right]+\frac{1}{3b}\left[c^2\cos A+a^2\cos C\right.$$

$$\left.+\frac{4\Delta^2}{b^2}\log\left(\frac{a+b+c}{a+c-b}\right)\right]+\frac{1}{3c}\left(a^2\cos B+b^2\cos A+\frac{4\Delta^2}{c^2}\log\left(\frac{a+b+c}{a+b-c}\right)\right],$$

where Δ=area of triangle.

$$\therefore \Delta=\tfrac{1}{3}[a+b+c]+\frac{1}{6a^2}[b+c][b-c]^2+\frac{1}{6b^2}[a+c][a-c]^2$$

$$+\frac{1}{6c^2}[a+b][a-b]^2+\frac{4\Delta^2}{3}\left[\frac{1}{a^3}\log\left(\frac{a+b+c}{b+c-a}\right)+\frac{1}{b^3}\log\left(\frac{a+b+c}{a+c-b}\right)\right.$$

$$\left.+\frac{1}{c^2}\log\left(\frac{a+b+c}{a+b-c}\right)\right].$$

If $a=b=c$, $\Delta=a[1+\frac{3}{4}\log 3]$.

II. Solution by HENRY HEATON, Belfield, N. D.

Let P be the point, and $AD=h$, the perpendicular from A upon BC. Put $AP=x$, and $\angle PAD=\theta$. Then the average length of AP is

$$\int_{B-\frac{1}{2}\pi}^{\frac{1}{2}\pi-C}\int_0^{h\sec\theta} x^2\,d\theta dx \div \int_{B-\frac{1}{2}\pi}^{\frac{1}{2}\pi-C}\int_0^{h\sec\theta} x\,d\theta dx = \frac{2h}{3}\int_{B-\frac{1}{2}\pi}^{\frac{1}{2}\pi-C}\sec^3\theta\,d\theta \div \int_{B-\frac{1}{2}\pi}^{\frac{1}{2}\pi-C}\sec^2\theta\,d\theta$$

$$=\frac{h}{3}\left(\cot C\operatorname{cosec} C+\cot B\operatorname{cosec} B-\log[\tan\tfrac{1}{2}C\tan\tfrac{1}{2}B]\div\cot C+\cot B\right.$$

$$=\frac{1}{3a}\left(b^2\cos C+c^2\cos B-bc\sin B\sin C\log[\tan\tfrac{1}{2}C\tan\tfrac{1}{2}B].\right.$$

In like manner it may be shown that the average length of BP is

$$\frac{1}{3b}\left(a^2\cos C+c^2\cos A-ac\sin A\sin C\log[\tan\tfrac{1}{2}A\tan\tfrac{1}{2}C]\right.; \text{ and of } CP,$$

$$\frac{1}{3c}\left(a^2\cos B+b^2\cos A-ab\sin A\sin B\log[\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B]\right.$$

Hence the required average is

$$M=\tfrac{1}{3}\left[\left(\frac{b^2}{c}+\frac{c^2}{b}\right)\cos A+\left(\frac{a^2}{c}+\frac{c^2}{a}\right)\cos B+\left(\frac{a^2}{b}+\frac{b^2}{a}\right)\cos C-a\sin\tfrac{1}{2}A[b\sin B+c\sin C]\log\tan\tfrac{1}{2}A-b\sin B[a\sin A+c\sin C]\log\tan\tfrac{1}{2}B-c\sin C[a\sin A+b\sin B]\log\tan\tfrac{1}{2}C.\right.$$

184. Proposed by HENRY HEATON, Belfield, N. D.

Through every point of the sides of a given square, straight lines are drawn across the square in every possible direction. What is their average length?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The problem evidently wants the average length of all lines terminated in opposite sides; otherwise the problem is the same as problem 169, three solutions of which have already been published.

Let $[a, x]$ be the coordinates of one end of the line, $[0, y]$ the coordinates of the other end. $\triangle$ =the average length required.

$$\therefore \triangle=\frac{\int_0^a\int_0^x\sqrt{\{a^2+[x-y]^2\}}\,dx\,dy}{\int_0^a\int_0^x dx dy}=\frac{2}{a^2}\int_0^a\int_0^x\sqrt{\{a^2+[x-y]^2\}}\,dx dy$$

$$=\frac{1}{a^2}\int_0^a\left(x\sqrt{[a^2+x^2]}+a^2\log\frac{x+\sqrt{[a^2+x^2]}}{a}\right)dx=a\{\tfrac{2}{3}[1-\sqrt{2}]+\log[1+\sqrt{2}]\}.$$

II. Solution by HENRY HEATON, Belfield, N. D.

Let P be a point in AB, PE a line perpendicular to AB, and PF a line drawn across the $\triangle PAD$. Put $AP=x$, and $\angle FPA=\theta$. Then supposing x constant, the average length of the lines drawn from P across the triangle PAD is

$$\int_0^{\tan^{-1}(a/x)} x\sec\theta\,d\theta\div\int_0^{\tan^{-1}(a/x)} d\theta=x\log\left(\frac{\sqrt{[a^2+x^2]}+a}{x}\right)\div\tan^{-1}[a/x].$$

Similarly the average length of lines drawn from P across the $\triangle PED$ is

$$a \log \left(\frac{\sqrt{[a^2+x^2]}+x}{a} \right) \div \tan^{-1}[x/a].$$

Hence the average length of lines drawn from P across the rectangle $PEDA$ is

$$\frac{2}{\pi}\left[x \log \left(\frac{\sqrt{[a^2+x^2]}+a}{x} \right) + a \log \left(\frac{\sqrt{[a^2+x^2]}+x}{a} \right) \right].$$

Hence the required average is

$$M = \frac{4}{a\pi} \int_0^a \left[x \log \left(\frac{\sqrt{[a^2+x^2]}+a}{x} \right) + a \log \left(\frac{\sqrt{[a^2+x^2]}+x}{a} \right) \right] dx$$

$$= \frac{6a}{\pi} \log[\sqrt{2}+1] - \frac{2}{\pi} \int_0^a \frac{x\,dx}{\sqrt{[a^2+x^2]}} = \frac{2a}{\pi} \{3 \log \sqrt{[2+1]} - \sqrt{[2+1]}\}.$$

NOTE.—These solutions differ because both problems are stated in the indefinite form and the authors have assumed different laws of distribution. ED. F.

MISCELLANEOUS.

168. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Sum to n terms, $\sin\alpha \sin\beta + \sin\alpha - \beta \sin\beta + \gamma + \sin\alpha - 2\beta \sin\beta + 2\gamma + \ldots$

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The series intended is evidently

$$\sin\alpha \sin\beta + \sin[\alpha-\beta]\sin[\beta+\gamma] + \sin[\alpha-2\beta]\sin[\beta+2\gamma] + \ldots \text{ to } n \text{ terms.}$$

Since $2\sin\alpha \sin\beta = \cos[\alpha-\beta] - \cos[\alpha+\beta]$ we get, if S=the sum

$$S = \tfrac{1}{2}\{\cos[\alpha-\beta] + \cos[\alpha-2\beta-\gamma] + \cos[\alpha-3\beta-2\gamma] + \ldots \text{ to } n \text{ terms}\}$$
$$-\tfrac{1}{2}\{\cos[\alpha+\beta] + \cos[\alpha+\gamma] + \cos[\alpha-\beta+\gamma] + \ldots \text{ to } n \text{ terms}\}.$$

Let $\alpha+\beta=\theta$, $=[\beta+\gamma]=\phi$, $[\alpha+\beta]=\psi$, $-[\beta-\gamma]=\rho$.

$$\therefore S = \tfrac{1}{2}\{\cos\theta + \cos[\theta+\phi] + \cos[\theta+2\phi] + \ldots + \text{ to } n \text{ terms}\}$$
$$-\tfrac{1}{2}\{\cos\psi + \cos[\psi+\rho] + \cos[\psi+2\rho] + \ldots + \text{ to } n \text{ terms}\}.$$

$$\therefore S=\frac{\cos[\theta+\frac{n-1}{2}\phi]\sin\frac{n\phi}{2}}{2\sin\frac{\phi}{2}}-\frac{\cos[\psi+\frac{n-1}{2}\rho]\sin\frac{n\rho}{2}}{2\sin\frac{\rho}{2}}$$

$$=\frac{\cos\{\alpha-\beta-\frac{n-1}{2}[\beta+\gamma]\}\sin\frac{n[\beta+\gamma]}{2}}{2\sin\frac{\beta+\gamma}{2}}-\frac{\cos\{\alpha+\beta-\frac{n-1}{2}[\beta-\gamma]\}\sin\frac{n[\beta-\gamma]}{2}}{2\sin\frac{\beta-\gamma}{2}}.$$

Also solved by A. H. Holmes.

PROBLEMS FOR SOLUTION.

GEOMETRY.

319. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Given the radii and the distances apart of the centers of three circles, to find the radii of the eight circles touching the three given circles.

MECHANICS.

204. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A set of particles have coplanar motion due to mutual attractions. Each particle is now affected with a velocity V parallel to a fixed direction. How will this affect the angular momentum of the set about their centroid?

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

147. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

If $4n+3$ is prime, $2(1, 2, 3, \ldots, 4n)+1\equiv 0 \pmod{4n+3}$; and conversely. If $4n+3$ is prime, $(1, 2, 3, \ldots, 2n)^2-4\equiv 0 \pmod{4n+3}$; and conversely.

AVERAGE AND PROBABILITY.

190. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

A line is drawn at random across a regular $2n$-gon; what is the chance that it crosses parallel sides?

MISCELLANEOUS.

172. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If ϕ and ψ are small angles, show that an approximate value of ϕ/ψ is

$$\tfrac{2}{3}\frac{\sin\phi}{\sin\psi}+\tfrac{1}{3}\frac{\tan\phi}{\tan\psi}-\tfrac{1}{180}(\phi^2-\psi^2)(9\phi^2-\psi^2).$$

NOTES AND NEWS.

The August-September number of THE AMERICAN MATHEMATICAL MONTHLY will not appear until September 28th.

Dr. A. L. Underhill, Instructor at Princeton University, has been appointed to an Instructorship in Mathematics at the University of Wisconsin.

Professor A. L. Rhoton, of Southwestern University, Jackson, Tenn., has been appointed Professor of Mathematics at Georgetown College, Georgetown, Kentucky.

Mr. W. V. Lovitt, who took the Master's degree at the University of Chicago in June, has been appointed to an Instructorship in Mathematics at the University of Washington, Seattle, Wash.

At a recent meeting of the regents of the University of West Virginia, Dr. John A. Eiesland, of the United States Naval Academy, was elected to the chair of mathematics in that institution.

Dr. Oswald Veblen, Preceptor in Princeton University, has been promoted to an Assistant Professorship in Mathematics. Mr. Veblen received his Doctor's degree at the University of Chicago in 1903.

Mr. G. D. Birkhoff received his Doctorate in Mathematics at the University of Chicago at the June Convocation. Dr. Birkhoff has been appointed Instructor in Mathematics at the University of Wisconsin.

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AN EXAMPLE OF THE INDICATRIX IN THE CALCULUS OF VARIATIONS.*

By ARNOLD DRESDEN, The University of Chicago.

(Continued.)

§ 5. *Case I:* $0<r<a$.

From our previous work, we obtain

$$F(r, \psi)>0, \quad F_1(r, \psi)>0, \quad 0 \leqq \psi \leqq 2\pi.$$

We conclude from this, by theorems *A* and *B* (see page 121), that the Indicatrix for any point in this region is a curve, which is closed around the origin G_1 and which has constantly positive curvature (see Fig. 5, curve *A*). It follows, that all the tangents lie entirely outside the curve, so that all points $\bar{Q}$ lie on the same side of the tangent at *any* point Q, as the origin G. Since, moreover, $F>0$, we have by theorem *C* (see page 122), that

$$E(x, y; \cos\theta, \sin\theta; \cos\bar{\theta}, \sin\bar{\theta})>0,$$

for all pairs $(\theta, \bar{\theta})$, $0 \leqq \theta \leqq 2\pi$, $0 \leqq \bar{\theta} \leqq 2\pi$, except when $\theta=\bar{\theta}$.

But this, by Nos. II and IV of our general theorems (see page 120) expresses the following result:

II. *Every extremal-element*† (x, y, ψ) *in the region I is an element of a hyperstrong minimum.*

This result could have been obtained more simply, by observing that since $F_1>0$, for all values of ψ between 0 and 2π, condition (IV′) is satisfied.

It follows further, that the curve does not admit double tangents, so that *no point in the region I can be the corner of a discontinuous solution,* which is evident by theorem *D* (page 123).

*For the first part of this paper, see the previous number of this journal.

†To say, that (x, y, ψ) is an extremal element is a short way of expressing, that it is possible to draw through the point (x, y) an extremal in a direction, which makes an angle ψ with the radius-vector at (x, y), and consequently an angle θ with the positive X-axis (compare p. 124, equation (9) and Fig. 2).